Quantum state conversion between continuous variable and qubits systems

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Abstract

We investigate how quantum state can be converted between continuous variable and qubits systems. Non-linear Jaynes-Cumings interaction Hamiltonian is introduced to accomplish the conversion. Detail analysis on the conversion of thermal state exhibits that pretty good fidelity can be achieved.

Keywords: quantum state conversion; non-linear Jaynes-Cumings model.

1 Introduction

Quantum information processing (QIP) has been extensively studied for a qubit system which is a quantum extension of a bit, spanning two-dimensional Hilbert space. A qubit is realized by an electronic spin, a two-level atom, the polarization of a photon and a superconductor among others. Parallelly, much attentions have been paid to the QIP of quantum continuous variable (CV) system, CV physical systems such as a harmonic oscillator, a rotator and a light field are defined in infinitive-dimensional Hilbert space. In contrast to the easily manipulation and storage of qubit system, CV system is more suitable for quantum information transformation. Until recently, qubit and CV systems are nearly always treated separately. The physical interface between CV and qubit system is waiting to be exploited. There have been some pilot works on how to convert quantum continuous variable to qubit system, but the efficient of the conversion is quite low[1]. We would propose a scheme of converting the quantum information between the two systems with high fidelity in this letter.

Quantum CV system is described by density operator ρ which is a function of the annihilation operators $a_1, \cdots, a_s$ and creation operators $a_1^\dagger, \cdots, a_s^\dagger$ satisfying canonical communication relation $[a_j, a_k^\dagger] = \delta_{jk}$.The subscripts of the annihilation or creation operator are for different modes (frequencies and polarizations of the optical state). A qubit is described by density operator of two-dimensional Hilbert space. Let the basis of the two-dimensional Hilbert space be $|-\rangle$ and $|+\rangle$ (i.e. lower and higher levers of two levels atomic system), the density operator is $\varrho = \alpha |-\rangle \langle -| + \beta |-\rangle \langle +| + \beta^* |+\rangle \langle -| + (1-\alpha) |+\rangle \langle +|$. In the following we will consider the quantum information conversion between single mode CV field and qubits system or vice via. The aim of the quantum state conversion is to convert CV quantum state to qubits system with high fidelity (or the reverse process), that is, we use many qubits to express the CV quantum state. It seems that this is much like quantum source coding[2], but they are very different. In quantum source coding of a CV state ρ', the aim is to convert $\rho'^{\otimes m}$ (many copies of ρ',i.i.d) to $\varrho'^{\otimes n}$ with least n/m at high fidelity. The qubit state ϱ' in quantum source coding is a state with entropy almost being equal to 1(thus the state ρ' is efficiently compressed). While in our quantum state conversion, the input state ρ is only one copy, and the serial of output states ϱ_i $(i = 1, \cdots, n)$ may not be identical and usually not be maximal entropy states.

2 The scheme of quantum state conversion

In quantum information theory, all logical operations on qubit as well as transformation on CV should be implemented with unitary transformation. The interaction among different systems gives rise to the transfer of quantum information between the systems. The scheme considered here is one mode field interact with each qubit of the qubit series consecutively, similar to that used in quantum information dilution[3]. Let the length of

the qubit series be K. The $k-th$ step of interaction has the form that the field and the $k-th$ qubit are prepared in the product state $\rho_k(0)\otimes\varrho_k(0)$, where $\varrho_k(0)$ is in the definite initial state $\left|-\right\rangle\left\langle-\right|$. The whole system will evolve in the way of $U_k(t_k)\,\rho_k(0)\otimes\varrho_k(0)\,U_k^+(t_k)$, where $U_k(t_k)=\exp(-\frac{i}{\hbar}H_kt_k)$ is the evolution operator in interaction picture, with H_k being the interaction Hamiltonian which characterizes the interaction between the field and the $k-th$ qubit. After the evolution for a time interval t_k, the field and the qubit may get entangled. At the end of the time interval, the interaction between the field and the $k-th$ qubit will be turned off. The field density operator now is $\rho_k(t_k)$, we set it as the initial state $\rho_{k+1}(0)$ of the next step, where time is reset to 0. Then the field is moved to interact with the $(k+1)-th$ qubit and leave the $k-th$ qubit alone. The former qubits should not be dropped. After the field interact with each qubit we drop the field, that is, we trace off the field density operator. The field state is transformed to that of a series of qubit states. Although the qubits are prepared in the same definite initial state $\left|-\right\rangle\left\langle-\right|$, and there are no direct interaction among qubits, the final state of the qubit system may be an entangled system via the field interacts with each qubit. The residue field state will tend to the vacuum state after the conversion as K tends to infinitive. We will elucidate this by an example in the next section.

3 Conversion of quantum thermal state

For simplicity, we choose quantum thermal state to elucidate the whole idea of quantum state conversion. The density operator of thermal state is $\rho=(1-v)\sum_{m=0}^{\infty}v^m\left|m\right\rangle\left\langle m\right|$, where $v=N/(N+1)$, with N being the average photon number of the state. $\left|m\right\rangle$ is the number state with photon number m, it is the eigenstate of particle number operator $\widehat{n}=a^+a$. Before the first step of conversion, the combined state is $\rho\otimes\left|-\right\rangle_{1,1}\left\langle-\right|$, the subscript 1 represents the first qubit. Suppose the interaction model Hamiltonian of CV field system and the atomic qubit system is

$$H_1=\hbar\Omega\left(\sqrt{\widehat{n}}a^+\sigma_-+a\sqrt{\widehat{n}}\sigma_+\right), \tag{1}$$

where σ_- and σ_+ are operators which convert the atom state form its excited state $\left|+\right\rangle$ to ground state $\left|-\right\rangle$ and from ground state to excited state respectively. $\hbar\Omega$ is the energy difference of the two energy levels. The Hamiltonian (1) can be considered as a kind of nonlinear Jaynes-Cummings model[4][5]. Then

$$\exp(-\frac{i}{\hbar}H_1t_1)\left|m,-\right\rangle_1=\cos(m\Omega t_1)\left|m,-\right\rangle_1-i\sin(m\Omega t_1)\left|m-1,+\right\rangle_1. \tag{2}$$

If the interaction time t_1 is adjusted in such a way that $\Omega t_1=\pi/2$, then

$$\begin{aligned}\exp(-\frac{i}{\hbar}H_1t_1)\left|2m,-\right\rangle_1 &= (-1)^m\left|2m,-\right\rangle_1, \\ \exp(-\frac{i}{\hbar}H_1t_1)\left|2m+1,-\right\rangle_1 &= -i(-1)^m\left|2m,+\right\rangle_1.\end{aligned} \tag{3}$$

Applying the evolution operator $U_1(t_1)=\exp(-\frac{i}{\hbar}H_1t_1)$ to the state $\rho\otimes\left|-\right\rangle_{1,1}\left\langle-\right|$, we have

$$U_1(t_1)\rho\otimes\left|-\right\rangle_{1,1}\left\langle-\right|U_1^{\dagger}(t_1)=(1-v)\sum_{m=0}^{\infty}v^{2m}\left|2m\right\rangle\left\langle 2m\right|\otimes\left(\left|-\right\rangle_{1,1}\left\langle-\right|+v\left|+\right\rangle_{1,1}\left\langle+\right|\right) \tag{4}$$

It should be noticed that the state after evolution is a product state of CV system state and qubit state, the field and the qubit become separable. Some of the quantum information has been already transferred to qubit system, this can be seen from the entropy of the qubit state. The first qubit state after the transfer is

$$\varrho_1=\frac{1}{1+v}\left(\left|-\right\rangle_{1,1}\left\langle-\right|+v\left|+\right\rangle_{1,1}\left\langle+\right|\right), \tag{5}$$

whose entropy is $S(\varrho_1)=\log_2(1+v)-\frac{v}{1+v}\log_2 v$. The density operator of the field after the evolution will be

$$\rho_2(0)=\rho_1(t_1)=(1-v^2)\sum_{m=0}^{\infty}v^{2m}\left|2m\right\rangle\left\langle 2m\right|. \tag{6}$$

The CV state $\rho_2(0)$ has even number of photons in each item. We can separate the first qubit state ϱ_1 from the combined state, then append the second qubit state which is prepared in definite initial state $|-\rangle_{2,2}\langle-|$ to the field, the new combined state will be $\rho_2(0)\otimes|-\rangle_{2,2}\langle-|$. We would design another interaction Hamiltonian to assign part of CV state to the second qubit. The Hamiltonian will be

$$H_2 = \hbar\Omega(\sqrt{\widehat{n}}a^+\frac{1}{\sqrt{\widehat{n}}}a^+\sigma_- + a\frac{1}{\sqrt{\widehat{n}}}a\sqrt{\widehat{n}}\sigma_+), \tag{7}$$

the evolution operator will be $U_2(t_2) = \exp(-\frac{i}{\hbar}H_2t_2)$. Then

$$\exp[-\frac{i}{\hbar}H_2t_2]\,|2m,-\rangle_2 = \cos(2m\Omega t_2)\,|2m,-\rangle_2 - i\sin(2m\Omega t_2)\,|2m-2,+\rangle_2\,. \tag{8}$$

If the interaction time t_2 is adjusted in such a way that $\Omega t_2 = \pi/4$, then

$$\begin{aligned}\exp(-\frac{i}{\hbar}H_2t_2)\,|4m,-\rangle_2 &= (-1)^m\,|4m,-\rangle_2\,, \\ \exp(-\frac{i}{\hbar}H_2t_2)\,|4m+2,-\rangle_2 &= -i(-1)^m\,|4m,+\rangle_2\,.\end{aligned} \tag{9}$$

Applying the evolution operator $U_2(t_2) = \exp(-\frac{i}{\hbar}H_2t_2)$ to the state $\rho_2(0)\otimes|-\rangle_{2,2}\langle-|$, we have

$$U_2(t_2)\rho_2(0)\otimes|-\rangle_{2,2}\langle-|U_2^\dagger(t_2) = (1-v^2)\sum_{m=0}^{\infty}v^{4m}\,|4m\rangle\langle 4m|\otimes(|-\rangle_{2,2}\langle-|+v^2\,|+\rangle_{2,2}\langle+|) \tag{10}$$

The state after evolution is a product state of CV system state and qubit state again. The second qubit state after the transfer is

$$\varrho_2 = \frac{1}{1+v^2}(|-\rangle_{2,2}\langle-|+v^2\,|+\rangle_{2,2}\langle+|), \tag{11}$$

whose entropy is $S(\varrho_2) = \log_2(1+v^2) - \frac{v^2}{1+v^2}\log_2 v^2$. The CV system after the evolution will be

$$\rho_3(0) = \rho_2(t_2) = (1-v^4)\sum_{m=0}^{\infty}v^{4m}\,|4m\rangle\langle 4m|\,. \tag{12}$$

The second qubit state ϱ_2 then is removed from the combined state, and third qubit state is appended onto the field. The $k-th$ Hamiltonian will be $H_k = \hbar\Omega[\widehat{n}(\frac{1}{\sqrt{\widehat{n}}}a^+)^{2^{k-1}}\sigma_- + (a\frac{1}{\sqrt{\widehat{n}}})^{2^{k-1}}\widehat{n}\sigma_+]$ and interaction time is $t_k = \pi/(2^k\Omega)$. After all the evolution the whole state will be

$$\begin{aligned}& U_K(t_K)\cdots U_2(t_2)U_1(t_1)\rho\otimes|-,-,\cdots,-\rangle\langle-,-,\cdots,-|\,U_1^\dagger(t_1)U_2^\dagger(t_2)\cdots U_K^\dagger(t_K) \\ =\ & (1-v^{2^K})\sum_{m=0}^{\infty}v^{2^K m}\,\left|2^K m\right\rangle\left\langle 2^K m\right|\otimes\varrho,\end{aligned} \tag{13}$$

where $\varrho = \varrho_1\otimes\varrho_2\otimes\cdots\otimes\varrho_K$ with $\varrho_k = (|-\rangle_{k,k}\langle-|+v^{2^{k-1}}\,|+\rangle_{k,k}\langle+|)/(1+v^{2^{k-1}})$. The information transferred to $k-th$ qubit is $S(\varrho_k) = \log_2(1+v^{2^{k-1}}) - \frac{v^{2^{k-1}}}{1+v^{2^{k-1}}}\log_2 v^{2^{k-1}}$. The total information transferred is

$$\sum_{k=1}^{K}S\left(\varrho_k\right) = S(\rho) - S[\rho_K(t_K)]. \tag{14}$$

The residue state of the field after all the evolution is $\rho_K(t_K) = (1-v^{2^K})\sum_{m=0}^{\infty}v^{2^K m}\left|2^K m\right\rangle\left\langle 2^K m\right|$, with its entropy being

$$S[\rho_K(t_K)] = -\log_2(1-v^{2^K}) - \frac{2^K v^{2^K}}{1-v^{2^K}}\log_2 v. \tag{15}$$

The residue CV state then is traced. The entropy loss is $S[\rho_K(t_K)]$ in this quantum state conversion procedure. In each step of evolution the entropy of the combined state is preserved, this is because unitary operation does not change the entropy of the combined state. The entropy loss can only occur at the last step of dropping the residue CV system. When $K \to \infty$, we have $S[\rho_K(t_K)] \to 0$, the entropy transferred to the qubit system tends to $S(\rho)$. The information is perfectly transferred. While for finite K, from the qubit series state $\varrho_1, \varrho_2, \cdots, \varrho_K$, the reconstructed CV state (see Section 5) is

$$\rho_r = \frac{1-v}{1-v^{2^K}} \sum_{m=0}^{2^K-1} v^m |m\rangle \langle m| \,. \tag{16}$$

The fidelity of the state conversion is

$$F = Tr(\sqrt{\sqrt{\rho}\rho_r\sqrt{\rho}}) = \sqrt{1-\nu^{2^K}}. \tag{17}$$

Other measure of the state conversion fidelity is the closeness of the residue field to the vacuum filed state, which is

$$F' = \sqrt{\langle 0| \rho_K(t_K) |0\rangle} = \sqrt{1-\nu^{2^K}}. \tag{18}$$

The two fidelities are equal in quantum thermal state conversion. For most of the thermal states ($\nu \nrightarrow 1$), pretty good fidelity can be achieved by using several qubits.

4 General quantum state conversion

In this section we will consider quantum state conversion of a general single mode quantum state. The initial CV state is $\rho = \sum_{n,m=0}^{\infty} c_{nm} |n\rangle \langle m|$, the first step of conversion will be

$$\begin{aligned} \exp[-\frac{i}{\hbar}H_1 t_1]\rho \otimes |-\rangle_{1,1} \langle -| \exp[\frac{i}{\hbar}H_1 t_1] \quad &= \quad \sum_{n,m=0}^{\infty} (-1)^{m+n} |2n\rangle \langle 2m| (c_{2n,2m} |-\rangle_{1,1} \langle -| + ic_{2n,2m+1} |-\rangle_{1,1} \langle +| \\ &\quad -ic_{2n+1,2m} |+\rangle_{1,1} \langle -| + c_{2n+1,2m+1} |+\rangle_{1,1} \langle +|). \end{aligned} \tag{19}$$

Then $|-\rangle_{2,2} \langle -|$ is appended and the unitary transformation $\exp[-\frac{i}{\hbar}H_2 t_2]$ is applied and so on, at last $|-\rangle_{K,K} \langle -|$ is appended and $\exp[-\frac{i}{\hbar}H_K t_K]$ is applied, each item of the CV part will convert to a form of $|2^K n\rangle \langle 2^K m|$. At this stage the entropy of the combined state remains intact. We obtain the qubit series by tracing the residue field. Denote $j = 2^K n + 2^{K-1} j_K + 2^{K-2} j_{K-1} + \cdots + j_1 = 2^K n + (j_K j_{K-1} \cdots j_1)$, with $j_k = 0, 1$;and $|-\rangle_k = |j_k = 0\rangle, |+\rangle_k = |j_k = 1\rangle$, then

$$\begin{aligned} & U_K(t_K)\cdots U_2(t_2)U_1(t_1)\rho \otimes |-,-,\cdots,-\rangle \langle -,-,\cdots,-| U_1^\dagger(t_1)U_2^\dagger(t_2)\cdots U_K^\dagger(t_K) \\ = \quad & \sum_{n,m=0}^{\infty} (-1)^{m+n} |2^K n\rangle \langle 2^K m| \otimes \sum_{j_K,j_{K-1},\cdots,j_1,l_K,l_{K-1},\cdots,l_1=0,}^{1} c_{2^K n+(j_K j_{K-1}\cdots j_1),2^K m+(l_K l_{K-1}\cdots l_1)} \\ & (-1)^{j_1+l_1} i^{j_K+j_{K-1}+\cdots+j_1} (-i)^{l_K+l_{K-1}+\cdots+l_1} |j_K j_{K-1}\cdots j_1\rangle \langle l_K l_{K-1}\cdots l_1| \,. \end{aligned} \tag{20}$$

After tracing the residue field, the result $K-$qubit state will be

$$\begin{aligned} \varrho \quad = \quad & \sum_{m=0}^{\infty} \sum_{j_K,j_{K-1},\cdots,j_1,l_K,l_{K-1},\cdots,l_1=0,}^{1} c_{2^K m+(j_K j_{K-1}\cdots j_1),2^K m+(l_K l_{K-1}\cdots l_1)} \\ & (-1)^{j_1+l_1} i^{j_K+j_{K-1}+\cdots+j_1} (-i)^{l_K+l_{K-1}+\cdots+l_1} |j_K j_{K-1}\cdots j_1\rangle \langle l_K l_{K-1}\cdots l_1| \,. \end{aligned} \tag{21}$$

By the reverse conversion (see the next Section), the reconstruct field state will be $\rho_r = \sum_{n,m=0}^{2^K-1} c'_{nm} |n\rangle \langle m|$, with $c'_{nm} = \sum_{n',m'=0}^{\infty} c_{2^K n'+n,2^K m'+m}$. When $K \to \infty$, we have $c'_{nm} \to c_{nm}$, the fidelity $F \to 1$. The fidelity may

not be easily calculated for a general input field state. Alternatively, we can calculate the closeness of the residue field ($\rho_{residue}$) with respect to the vacuum state, which turns out to be $F' = \sqrt{\langle 0|\rho_{residue}|0\rangle} = \sqrt{\sum_{j=0}^{2^K-1} c_{jj}}$. The residue field state is obtained by tracing all qubit freedoms of the last stage composed state. When the initial CV state is a coherent state $|\alpha\rangle = \exp(-|\alpha|^2/2)\alpha^n/\sqrt{n!}\,|n\rangle$, we have $F' = \sqrt{\exp(-|\alpha|^2)\sum_{j=0}^{2^K-1}|\alpha|^{2j}/j!}$.

5 Reverse conversion

In the reverse conversion, we have the initial state $\varrho_1 \otimes \varrho_2 \otimes \cdots \otimes \varrho_K$, where ϱ_k is the density operator of $k-th$ qubit. The process of reverse state conversion is to convert firstly the highest qubit ($K-th$) to the CV state then the lower. The combined state will evolve to

$$U_1^\dagger(t_1)\{\{U_2^\dagger(t_2)\cdots\{U_K^\dagger(t_K)|0\rangle\langle 0|\otimes\varrho_K U_K(t_K)\}\cdots\otimes\varrho_2 U_2(t_2)\otimes\varrho_1\}U_1(t_1), \tag{22}$$

where $|0\rangle\langle 0|$ is the initial state of the field. The first step is to transfer the state $\varrho_K = \alpha_K|-\rangle_{K,K}\langle -| + \beta_K|-\rangle_{K,K}\langle +| + \beta_K^*|+\rangle_{K,K}\langle -| + (1-\alpha_K)|+\rangle_{K,K}\langle +|$ to the field. Since

$$\begin{aligned}
\exp(\frac{i}{\hbar}H_K t_K)|0,-\rangle_K &= |0,-\rangle_K \\
\exp(\frac{i}{\hbar}H_K t_K)|0,+\rangle_K &= \cos(2^{K-1}\Omega t_K)|0,+\rangle_K + i\sin(2^{K-1}\Omega t_K)\left|2^{K-1},-\right\rangle_K .
\end{aligned} \tag{23}$$

The evolution time is so chosen that $\cos(2^{K-1}\Omega t_K) = 0$, we choose $2^{K-1}\Omega t_K = \pi/2$ as before. Then $\exp(\frac{i}{\hbar}H_K t_K)|0,+\rangle_K = i\left|2^{K-1},-\right\rangle_K$. The first step evolution will be$U_K^\dagger(t_K)\varrho_K \otimes |0\rangle\langle 0|U_K(t_K) = \rho_K \otimes |-\rangle_{K,K}\langle -|$, with

$$\rho_K = \alpha_K|0\rangle\langle 0| - i\beta_K|0\rangle\left\langle 2^{K-1}\right| + i\beta_K^*\left|2^{K-1}\right\rangle\langle 0| + (1-\alpha_K)\left|2^{K-1}\right\rangle\left\langle 2^{K-1}\right| \tag{24}$$

We see that all the information of qubit state ϱ_K is transferred to the field, leave the qubit state a definite blank state. Moreover, the combined state is a direct product of the field and qubit state, thus the $K-th$ qubit can be dropped after the evolution. The next step is to transfer ϱ_{K-1} to the remained field ρ_K. Since when $2^{K-2}\Omega t_{K-1} = \pi/2$, we have $U_{K-1}^\dagger(t_{K-1})|0,-\rangle_{K-1} = |0,-\rangle_{K-1}$, $U_{K-1}^\dagger(t_{K-1})\left|2^{K-1},-\right\rangle_{K-1} = -\left|2^{K-1},-\right\rangle_{K-1}$, $U_{K-1}^\dagger(t_{K-1})|0,+\rangle_{K-1} = i\left|2^{K-2},-\right\rangle_{K-1}$, $U_{K-1}^\dagger(t_{K-1})\left|2^{K-1},+\right\rangle_{K-1} = -i\left|2^{K-1}+2^{K-2},-\right\rangle_{K-1}$. The state after the evolution will be $\rho_{K-1}\otimes|-\rangle_{K-1,K-1}\langle -|\otimes|-\rangle_{K,K}\langle -|$. The quantum information of the two qubits are transferred to ρ_{K-1}. When all the qubits are transferred to the field, we get at last a quantum CV state $\rho = \rho_1$ while leaving all the qubit series in the lower energy level state $|-,-,\cdots,-\rangle$. Thus the reverse conversion procedure will convert a general qubit pair product state $\varrho_1\otimes\varrho_2\otimes\cdots\otimes\varrho_K$ into a continuous variable state ρ while keeping the entropy of the whole state. The reverse conversion is perfect.

6 Conclusions

The scheme of quantum state conversion between continuous variable and qubit systems has been proposed based on the unitary evolution. The interaction Hamiltonian is not a usual atom-photon (or electron-photon) interaction Hamiltonian. It is a kind of nonlinear Jaynes-Cummings model. Although it is physical, the realization of such a conversion is rather complicate and difficult. Theoretically, the conversion is asymptotically perfect. After the conversion, the field state is very close to vacuum, thus the field state is transferred to the state of qubit series. The scheme of state conversion can also be extended to the entanglement conversion between qubit system and continuous variable system[6].

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